Semicontinuity.

Sometimes continuity is too much to ask, and measurability too little.

Now, X is any topological space:

Def: ① Let $f: X \to [-\infty, \infty)$. If $\{x : f(x) < \alpha\}$ is open for all $\alpha \in \mathbb{R}$, then f is upper semicontinuous (USC).

② Let $f: (-\infty, \infty]$. If $\{x : f(x) > \alpha\}$ is open for all $\alpha \in \mathbb{R}$, then f is lower semicontinuous (LSC)

Rmks. ① Clearly, $f: X \to (-\infty, \infty)$ is cont. $\Leftrightarrow$ f is LSC & USC.

$$\left(\{x : a < f(x) < b\} = \{x : f(x) < b\} \cap \{x : f(x) > a\} \right)$$

(2) <u>USC</u> fcns are particularly important in complex analysis (+ potential theory, SCV..). The correct (best) setting for <u>subharmonic fcns</u> is that of USC fcns. In Conway (Math 220), full continuity is assumed, which is not optimal and leads to technical difficulties.

<u>Prop 1 (basic prop's of LSC fcns)</u>.

(a) U open $\Rightarrow \chi_U$ LSC.

(b) Let f be LSC and $c \in \mathbb{R}$. Then,
$c \geq 0 \Rightarrow cf$ LSC; $c \leq 0 \Rightarrow cf$ USC

(c) If $\mathcal{G}$ is any family of LSC fcns, then $f(x) = \sup\{g(x) : g \in \mathcal{G}\}$ is LSC.

(d) f_1, f_2 LSC $\Rightarrow f_1+f_2$ LSC

(e) Assume X LCH. f LSC & nonnegative

$$f(x) = \sup\{g(x) : g\in C_c(X), 0\le g\le f\}.$$

Rmks. (1) Folland considers only LSC fcns and mentions that similar results hold for USC fcns. For a cplx analyst (like me), USC fcns, as mentioned above, are more abundant (natural) than LSC ones. The way to transition from USC to LSC (or vice versa) is to note that if f is USC, then $-f$ is LSC (and v.v.)

(2) Observe that in (e) nonnegativity can be replaced by bdd from below. Indeed if $c \le f(x)$, $\forall x$, for some $c\in\mathbb{R}$ then $f(x)-c$ is nonnegative and still LSC. In the sup, of course, we then consider $c\le g\le f$.

<u>Pf of Prop 1.</u> DIY, or consult Folland Prop 7.11.

<u>Thm 1.</u> If X LCH, μ Radon, f nonneg. and LSC, then

$$\int f\,d\mu = \sup\left\{ \int g\,d\mu : g \in C_c(X),\ 0 \le g \le f \right\}.$$

<u>Pf.</u> We prove the following Prop.

<u>Prop 2.</u> If $\mathcal{G}$ is a family of nonneg. LSC fcns on X and $\forall g_1, g_2 \in \mathcal{G}$ $\exists g \in \mathcal{G}$ s.t. $g_1 \le g$ and $g_2 \le g$, then for any Radon measure μ:

$$\int f\,d\mu = \sup\left\{ \int g\,d\mu : g \in \mathcal{G} \right\}$$

It is immediate from Prop 1 (e) that $\int f\,d\mu \geq \sup\{\int g\,d\mu : g \in \mathcal{G}\}$. To prove the reverse ineq. consider the simple functions

$$\varphi_n = \frac{1}{2^n}\sum_{j=1}^{2^{2n}} \chi_{U_{j,n}};\quad \underbrace{U_{j,n} = \left\{x : f(x) > \frac{j}{2^n}\right\}}_{\text{open by LSC}}$$

Then, $\varphi_n \nearrow f$ a.e. (μ)

(cf. pf. of Thm 2.10 (a)).

By Monotone Conv. Thm, $\int \varphi_n \nearrow \int f$.

Pick $\varepsilon > 0$. Then $\exists N$ s.t. $\int \varphi_N > \int f - \varepsilon$

For this N, choose (possible by inner reg. on open sets) cpct $K_j \subseteq U_{j,N}$ s.t. $\psi = \frac{1}{2^N}\sum_1^{2^{2N}} \chi_{K_j}$ satisfies

$$\int \psi = \frac{1}{2^N}\sum \mu(K_j) > \int f - \varepsilon$$

(Recall $\int \varphi_N = \frac{1}{2^N}\sum_1^{2^{2N}} \mu(U_j) > \int f - \varepsilon$).

For each $x \in \bigcup_{j=1}^{2^{2N}} K_j$, $\swarrow$ cpct we have

$$\psi(x) \leq \varphi_N(x) < f(x) \overset{\text{def of } f}{\Rightarrow} \exists\, g_x \in \mathcal{G} \text{ s.t.}$$

$\psi(x) < g_x(x)$. Now, $g_x - \psi$ is LSC

(ψ is USC; χ_F for any closed F is USC)

$\Rightarrow$ $V_x = \underbrace{\{y : g_x(y) > \psi(y)\}}_{\{g_x - \psi > 0\}}$ is open and

$\bigcup_{j=1}^{2^{2N}} K_j \subseteq \bigcup_{x \in \bigcup K_j} V_x \leftarrow$ open cover of $\bigcup K_j$

Since $\bigcup K_j$ is cpct $\exists$ finite subcover

$\bigcup K_j \subseteq \bigcup_{l=1}^{k} V_{x_l}$. Choose $g \in \mathcal{G}$ s.t.

$g \geq g_{x_l}$, $l = 1, \dots, k$ $\Rightarrow$ for any

x s.t. $\psi(x) \neq 0$, $x \in \bigcup K_j \subseteq \bigcup_{1}^{k} V_{x_l}$

$\Rightarrow x \in V_{x_l}$ for some l $\Rightarrow$

$$\psi(x) < g_{x_l}(x) \leq g(x) \Rightarrow \psi < g$$

$\Rightarrow \int \varphi \leq \int g$. But $\int \varphi > \int f - \varepsilon$

$\Rightarrow \int g \geq \int \varphi > \int f - \varepsilon$. $\Rightarrow \forall \varepsilon > 0$,

$\exists g \in \mathcal{G}$ s.t. $\int g > \int f - \varepsilon \Rightarrow$

$$\int f \leq \sup \{ \int g : g \in \mathcal{G} \},$$

which is the desired reverse ineq. ∎

<u>Pf. of Thm 1.</u> We note that $\mathcal{G} := \{ g \in C_c(X) : 0 \leq g \leq f \}$ satisfies the assumption in Prop 2; for if $g_1, g_2 \in \mathcal{G}$ then $\max(g_1, g_2) \in \mathcal{G}$. By Prop 1 (e), $f(x) = \sup \{ g(x) : g \in \mathcal{G} \}$, so the conclusion of Thm 1 follows directly from Prop 2. ∎

<u>Thm 2</u> If μ is Radon measure, $f \geq 0$ and Borel measurable, then (i):

$$\int f\,d\mu = \inf\left\{\int g\,d\mu : g \geq f,\ g\ \text{LSC}\right\}$$

(ii) If $\{x : f(x) > 0\}$ is σ-finite, then

$$\int f\,d\mu = \sup\left\{\int g\,d\mu : 0 \leq g \leq f,\ g\ \text{USC}\right\}$$

<u>Pf.</u> See Folland Prop 7.14.